J. Fluid Mech. (1993), *vol.* 252, *pp.* 31–49

On the dispersion relation for trapped internal waves

By B. C. BARBER

Space Sector, Defence Research Agency, Farnborough, Hants, GU14 6TD, UK

(Received 13 September 1990 and in revised form 19 September 1992)

An analysis is constructed in order to estimate the dispersion relation for internal waves trapped in a layer and propagating linearly in a fluid of infinite depth with a rigid surface. The main interest is in predicting the structure of internal wave wakes, but the results are applicable to any internal waves. It is demonstrated that, in general $1/c_p = 1/c_{p0} + k/\omega_{\max} + \varepsilon(k)$ where c_p is the wave phase speed for a particular mode, c_{p0} is the phase speed at $k = 0$, $\omega_{\max}$ is the maximum possible wave angular frequency and $\omega_{\max} \leqslant N_{\max}$ where $N_{\max}$ is the maximum buoyancy frequency. Also, $\varepsilon(0) = 0$, $\varepsilon(k) = o(k)$ for k large, and is bounded for finite k. In particular, when $\varepsilon(k)$ can be neglected, the dispersion relation for a lowest mode wave is approximately $1/c_p \approx \left(\int_0^\infty N^2(y)y\,\mathrm{d}y\right)^{-\frac{1}{2}} + k/\omega_{\max}$. The eigenvalue problem is analysed for a class of buoyancy frequency squared functions $N^2(x)$ which is taken to be a class of real-valued functions of a real variable x where $0 \leqslant x \leqslant \infty$ such that $N^2(x) = O(\mathrm{e}^{-\beta x})$ as $x \rightarrow \infty$ and $1/\beta$ is an arbitrary length scale. It is demonstrated that $N^2(x)$ can be represented by a power series in $\mathrm{e}^{-\beta x}$. The eigenfunction equation is constructed for such a function and it is shown that there are two cases of the equation which have solutions in terms of known functions (Bessel functions and confluent hypergeometric functions). For these two cases it is shown that $\varepsilon(k)$ can be neglected and that, in addition, $\omega_{\max} = N_{\max}$. More generally, it is demonstrated that when $k \rightarrow \infty$ it is possible to approximate the equation uniformly in such a way that it can be compared with the confluent hypergeometric equation. The eigenvalues are then, approximately, zeros of the Whittaker functions. The main result which follows from this approach is that if $N^2(x)$ is $O(\mathrm{e}^{-\beta x})$ as $x \rightarrow \infty$ and has a maximum value $N^2_{\max}$ then a sufficient condition for $1/c_p \sim k/N_{\max}$ to hold for large k for the lowest mode is that $N^2(t)/t$ is convex for $0 \leqslant t \leqslant 1$ where $t = \mathrm{e}^{-\beta x}$.

1. Introduction

This paper describes work carried out in order to construct an analysis to estimate the dispersion relation for internal waves trapped in a layer. Some of the background to this analysis, including experimental and numerical work, is briefly described in this Introduction.

The requirement for the analysis described in this paper came about as a result of attempting to predict the shape and structure of internal ship wakes. Figure 1 shows a photograph of a 1500 ton ship (the RMAS Roysterer) in Loch Linnhe, Scotland on September 15, 1987 at 15:36:03 GMT. The photograph was taken from a helicopter at a height of 500 ft. The ship is at the lower left of the photograph and is steaming from top to bottom at a measured speed of 1.00 m/s relative to the land in water of about 120 m depth. It will be observed that there are bands on the water surface

FIGURE 1. The surface effect of an internal wave wake generated by RMAS Roysterer in Loch Linnhe.

extending diagonally across to the moored instrument platforms in the upper right of the photograph. These bands are the surface manifestation of a (lowest mode) internal wave wake trapped in a layer, and are caused by the long internal waves (with wavelengths varying between 80 and 250 m) interacting with the short surface waves. The water depth at the instrumented site was about 70 m.

An explanation of the modal structure of internal waves and the long wave–short wave interaction is given in Phillips (1977). This experiment was carried out at slack water and the tidal currents as measured at the instrument platforms were always less than 5 cm/s during this particular experiment and typically less than 2 cm/s. The wind speed and direction were about 2.5 m/s and 240° (the wind was blowing in the same direction as the ship velocity).

The patterns in figure 1 are qualitatively similar to the sketches presented in plate 4 of Eckman (1906).

Figure 2 shows a radar image corresponding to figure 1 and obtained 6 s after it. The radar image was obtained by an airborne synthetic aperture radar at a radar wavelength of 5.6 cm and with horizontal polarization at an incidence angle of 17.0° from the vertical at the ship. The image is in an orthographic map projection and corresponds to an area measuring approximately 3 km by 2 km. The spatial resolution was of the order of 5 m. The image in figure 2 has been shown to correspond closely with direct measurements of the (mode 1) internal waves made by conductivity–temperature–depth sensors (Perry 1992).

Clearly, from the point of view of pattern recognition the shape of wake images such as figure 2 is important. The structure of internal wake images is dependent on the dispersion relation (Keller & Munk 1970; Lighthill 1978). In order to predict the details of the pattern it is desirable to have a dispersion relation which is as simple as possible and depends on as few parameters as possible. The analysis reported in this

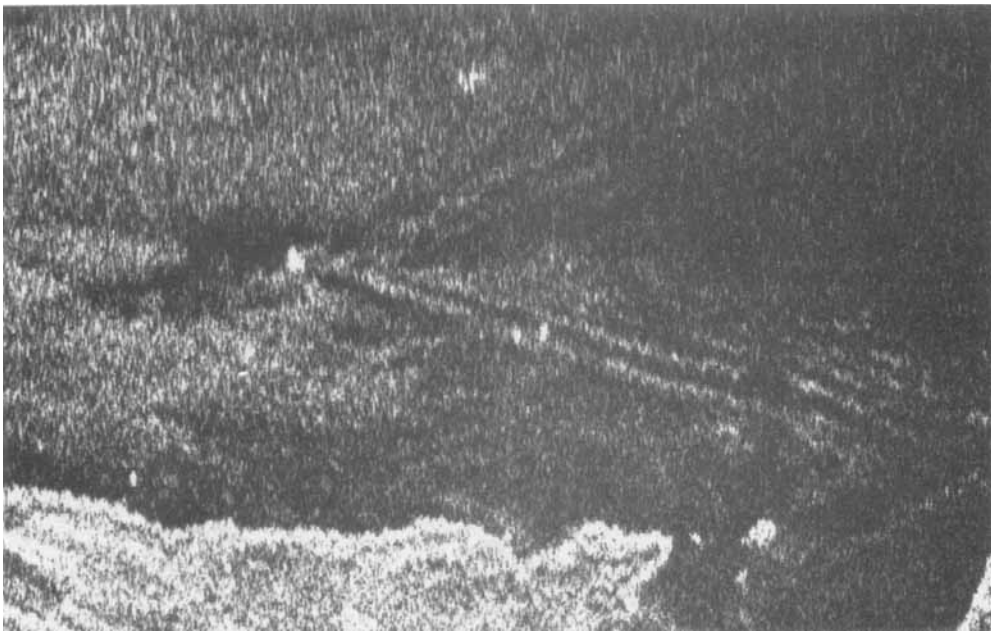

FIGURE 2. Radar image of the scene shown in figure 1.

paper was carried out in order to demonstrate the existence of a simple (approximate) two-parameter dispersion relation for trapped internal waves.

In the preceeding discussion it was implicitly assumed that the internal waves propagate linearly, and this point is now considered further. A typical maximum long-wave phase speed for internal waves is of the order of 1 m/s (in the case of the experiment described above it was about 50 cm/s). This is less than typical ship speeds and consequently a 'Mach wedge' usually exists. The 'Mach wedge' is here taken to be the neighbourhood of the leading edge of the wake system beyond which the waves do not propagate. In recent work (Miloh & Tulin 1988; Lee-Bapty 1991, 1992) it has been shown that for a two-layer fluid the interior of the wake is dominated by linear dispersion away from the Mach wedge. In the neighbourhood of the Mach wedge long waves dominate and the effect of dispersion is less. Consequently, nonlinear effects are important only in the region of the Mach wedge. However, even in the interior away from the Mach wedge, the cumulative effects of nonlinearity will eventually compete with the linear dispersion over long enough distances. Nevertheless, the dispersive terms in the wave equation dominate the nonlinear terms for many wavelengths behind the generating source and therefore the waves can be considered to propagate linearly in the wake interior. This argument is supported by experimental observations in the case of continuous stratification in a sea loch with a single main peak in the buoyancy frequency function (Watson, Chapman & Apel 1992; Stapleton & Perry 1992).

Generally, it is found experimentally that for the conditions described here the most visible surface effects are produced by the lowest mode (called mode 1 in this paper). Hence the analysis in this paper emphasizes the lower modes, especially mode 1.

The eigenfunction equation for trapped internal waves in the Boussinesq approximation is well known (Phillips 1977; Krauss 1966)

$$\psi_m''(x) + \left(\frac{N^2(x)}{c_p^2(k)} - k^2 \right) \psi_m(x) = 0. \tag{1.1}$$

The effects of the Earth's rotation have been ignored here, as have the effects of shear. The wave function ψ_m is the local wave amplitude, m is the mode number (where the lowest mode corresponds to $m = 1$), k is the horizontal wavenumber, c_p is the horizontal phase speed for a given mode, and $N^2(x)$ is the square of the buoyancy or Brunt–Väisälä frequency

$$N^2(x) = \frac{g}{\rho} \frac{\partial \rho}{\partial x}. \tag{1.2}$$

The direction of increasing depth is taken as the positive x-direction. Equation (1.1) together with two boundary conditions (e.g. $\psi(0) = 0$ and $\psi(x) \to 0$ as $x \to \infty$) constitute a Stürm–Liouville problem. The eigenvalues corresponding to the various modes are defined as $1/c_p$ throughout this paper. A rigid top boundary is assumed throughout and this amounts to an assumption that the frequency of the internal waves is very much less than that of any free surface waves; a very good approximation (Phillips 1977, equation 5.2.13).

Analysis of the Stürm-Liouville problem often involves three different types of sub–domain; the oscillatory, non-oscillatory, and transition (connecting) regions. Solutions valid in each sub-domain are then matched and patched together to produce the complete eigenfunction. An alternative approach is to seek approximate eigenfunctions which are uniformly valid over the whole x domain; this is the approach adopted in this paper. There is then no neccessity to analyse each domain separately and then hopefully the eigenvalues are simpler to estimate. Furthermore, as will become clear, the type of result sought in this analysis is of a very delicate kind which is intimately connected with the properties of the zeros of higher transcendental functions.

The ultimate objective of the work presented here is to estimate the dispersion relation for internal waves satisfying equation (1.1) given the density function $\rho(x)$ and the boundary conditions. Previous analytical attempts at this problem have involved a specific function (a 'model function') for $N^2(x)$, and results for several such model functions exist (Krauss 1966). In this paper an attempt is made to analyse the problem for a class of $N^2(x)$ functions. The class has to be limited in some way in order to have some chance of constructing a tractable analysis and so $N^2(x)$ is taken to be a class of real-valued functions of a real variable x where $0 \leqslant x \leqslant \infty$ such that $N^2(x) = O(e^{-\beta x})$ as $x \to \infty$. This seems to be the most promising way of circumscribing $N^2(x)$, and this type of functional behaviour has been considered before (Garrett & Munk 1972).

Before proceeding further some experimental data are discussed since these data will be used as an illustration. Figure 3 shows some density measurements made in Loch Linnhe, Scotland, on September 15, 1987 at the instrumented site at the upper right in figure 1. The y-axis in this figure has units of σ_T and this has the usual meaning, that is, the water density $= 1000.0 + \sigma_T$ kg/m^3. The 398 data points in figure 3 are the aggregate of six yo-yo dips made over a period of six minutes by means of a conductivity–temperature–depth sensor. This period of six minutes spanned the time when the images in figure 1 and figure 2 were obtained. Although these data show a shallow pycnocline (which was in fact a halocline) the method developed here is applicable to other situations.

There are a number of numerical methods available for solving a differential

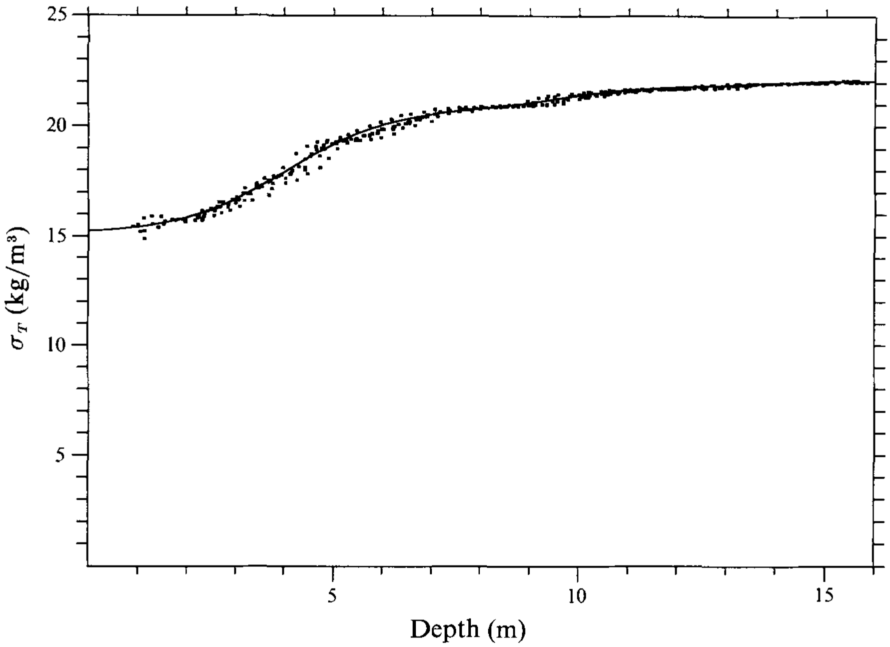

FIGURE 3. Density anomaly for the Loch Linnhe water column on 15 September 1987 during 15:33 to 15:39 GMT. Measured at RV Calanus on the instrumented site.

equation such as (1.1) and for this example a shooting method was selected (Press *et al.* 1986). The experimental data can also be modelled in a number of ways and a monotonically increasing function was fitted to the data using a nonlinear least squares technique. This curve is also shown in figure 3, and it fits the data well. The only difficulty which arises in the numerical solution of equation (1.1) in the case of a water column such as that shown in figure 3 is the usual one which results from the bottom boundary. Since $N^2(x) \to 0$ as $x \to \infty$ the equation has a solution $\psi \sim Ae^{-kx} + Be^{+kx}$ for large x. A numerical procedure thus has to implicitly cancel out a growing dominant exponential in the presence of the required (regressive) solution Ae^{-kx}. This was overcome in this example by choosing a finite depth in conjunction with a suitably large number of significant figures.

The computed dispersion curve for the case of the water column shown in figure 3 is shown in figures 4 and 5. Figure 4 shows a plot of ω against k for the first two modes and figure 5 shows a plot of $1/c_p$ against k for the same data. Notice that *the plot of* $1/c_p$ *against* k *is very nearly linear.* This is not an exceptional case. Many computations of dispersion relations with experimentally measured water columns like figure 3 have been carried out with very similar results. These computations lead to a particularly simple form for the dispersion relation

$$\omega = \frac{ck}{1+lk}, \quad \text{i.e.} \quad \frac{1}{c_p} = \frac{kl}{c} + \frac{1}{c}, \tag{1.3}$$

to a surprising degree of precision. In (1.3) c is an arbitrary wave phase speed and l is an arbitrary length scale. This approximate dispersion relation has proved to be very useful for predicting the patterns of ship-generated internal wave wakes (Perry 1992; Stapleton & Perry 1992; Watson, Chapman & Apel 1992).

Formula (1.3) is an interesting result which merits further investigation. Notice that

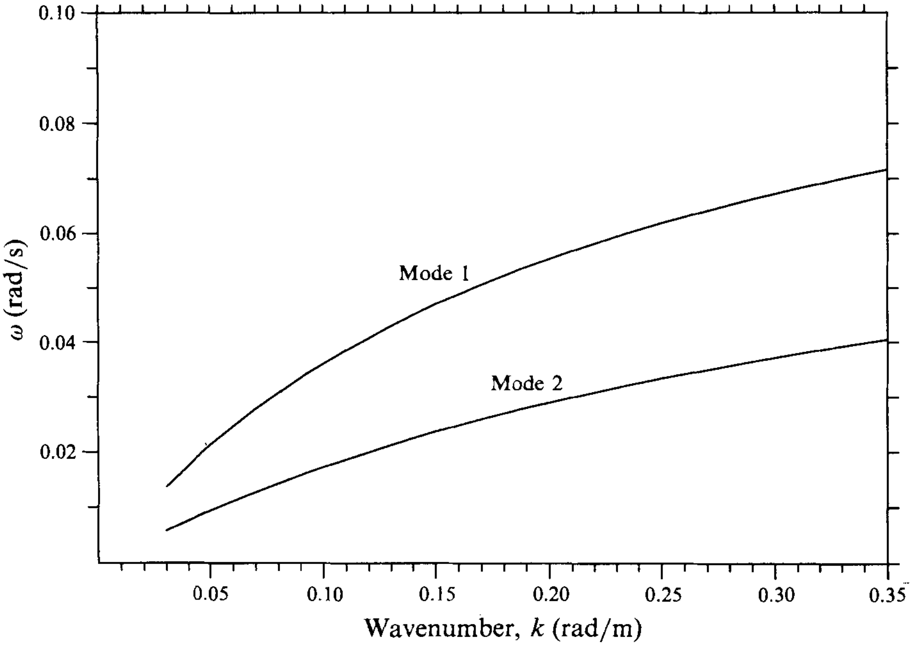

FIGURE 4. Internal wave dispersion relation for modes 1 and 2, computed from equations (1.1), (1.2) and the density function in figure 3.

figures 4 and 5 show that it is valid for wavelengths of the same order as the layer thickness and depth: This is not just a simple asymptotic result.

In the next section the general properties of the dispersion relation are described and it is shown that $1/c_p = 1/c_{p0} + k/\omega_{\max} + \varepsilon(k)$ where c_{p0} and $\omega_{\max}$ are constants to be defined, and $\varepsilon(k)$ is a function which is zero at $k = 0$. Also, $\varepsilon(k) = o(k)$ for k large, and is bounded for finite k. The main difficulty in verifying expression (1.3) analytically is in determining whether $\varepsilon(k)$ can be neglected, and this can only be done by considering specific cases. In §3 it is demonstrated that, when $N^2(x) = O(\mathrm{e}^{-\beta x})$ for large x, then $N^2(x)$ can be expanded as a power series in $\mathrm{e}^{-\beta x}$. The differential equation (1.1) is considered in §4 with an exponential power series for $N^2(x)$, and two special cases which lead to Bessel functions and confluent hypergeometric functions for the eigenfunctions are analysed in §§5 and 6. It is demonstrated that (1.3) is approximately valid for these two special cases. In §7 the general case is considered and a sufficient condition for the constant $\omega_{\max}$ to be equal to the maximum buoyancy frequency is worked out. Some effects of multiple peaks in the $N^2(x)$ function are described qualitatively in §8. Finally, the various results derived in the paper are summarized in §9.

2. General properties of the dispersion relation

2.1. *Small k*

It is easily demonstrated that the general solution of equation (1.1) in the form of an integral equation is

$$\psi(x) = A\mathrm{e}^{+kx} + B\mathrm{e}^{-kx} + \frac{1}{kc_p^2}\int_0^x N^2(y)\sinh k(y-x)\psi(y)\mathrm{d}y. \tag{2.1}$$

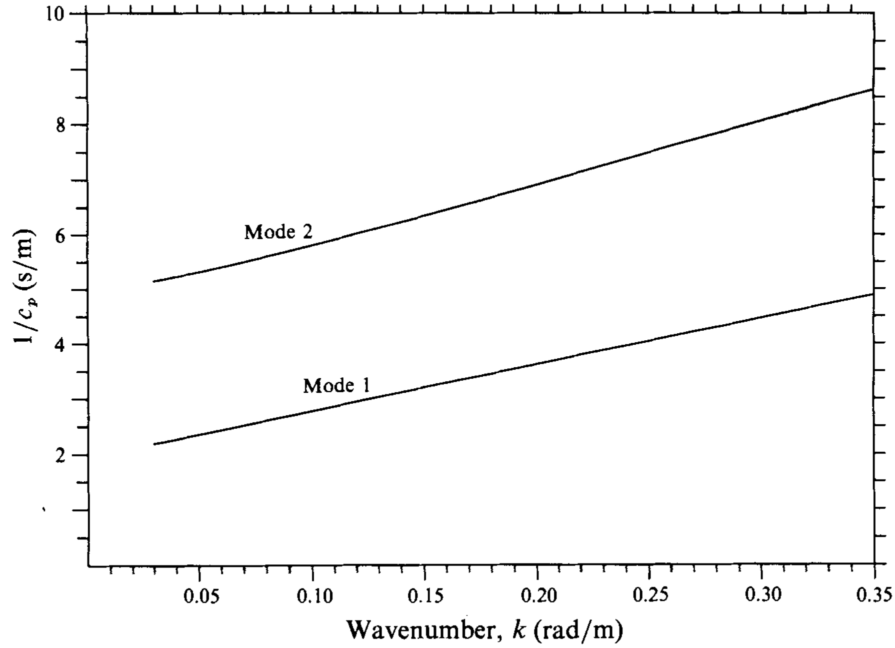

FIGURE 5. Internal wave dispersion relation for modes 1 and 2, computed from equations (1.1), (1.2) and the density function in figure 3.

Assume a rigid surface so that at $x = 0$, $\psi = 0$ and then $B = -A$ so that

$$\psi(x) = 2A\sinh kx + \frac{1}{kc_p^2}\int_0^x N^2(y)\sinh k(y-x)\psi(y)\mathrm{d}y. \tag{2.2}$$

$2A$ is arbitrary and it will be taken as unity so that at $x = 0$, $\psi'(x) = k$. As before, it is assumed that the water is very deep in comparison with $1/k$. At the bottom $\psi(x)$ is required to go to zero as $x \to \infty$, so those terms which grow exponentially at ∞ must sum to zero. Hence

$$0 = \frac{\mathrm{e}^{+kx}}{2} - \frac{\mathrm{e}^{+kx}}{2kc_p^2}\int_0^x N^2(y)\psi(y)\mathrm{e}^{-ky}\mathrm{d}y \tag{2.3}$$

as $x \to \infty$. Hence

$$\frac{1}{kc_p^2}\int_0^\infty N^2(y)\psi(y)\mathrm{e}^{-ky}\mathrm{d}y = 1. \tag{2.4}$$

This is also the requirement that $\psi'(x) \to 0$ as $x \to \infty$. This, coupled with the differential equation (1.1), means that all the derivatives go to zero at ∞. Hence $\psi(x) = O(x^n\mathrm{e}^{-kx})$ which is self-consistent since it has been implicitly assumed that $\mathrm{e}^{-kx}\int_0^x N^2(y)\psi(y)\mathrm{e}^{ky}\mathrm{d}y$ is bounded as $x \to \infty$.

Equation (2.2) is a Volterra equation and can be solved iteratively to give

$$\psi(x) = \sinh kx + \frac{1}{kc_p^2}\int_0^\infty N^2(y)\sinh k(y-x)\sinh ky\,\mathrm{d}y + \ldots, \tag{2.5}$$

where (2.5) is taken as far as the first iteration. Substituting (2.5) into (2.4) then gives

$$1 = \frac{1}{kc_p^2}\int_0^\infty N^2(y)\sinh ky\, e^{-ky}dy$$
$$+\frac{1}{k^2c_p^4}\int_0^\infty N^2(y)e^{-ky}dy\int_0^y N^2(z)\sinh k(y-z)\sinh kz\,dz+\dots . \qquad (2.6)$$

Continuing the iterations produces a power series in $1/c_p^2$ with an infinite number of roots corresponding to the phase speeds of the infinite set of modes (provided that $N^2(y)$ does not exclusively consist of a set of one or more delta functions). The lowest mode has the greatest long-wave phase speed (Phillips 1977, §5.2) and consequently this case corresponds to the smallest root of the power series in $1/c_p^2$. In such a case a first approximation to the smallest root is given by the first two terms

$$1 \approx \frac{1}{kc_p^2}\int_0^\infty N^2(y)\sinh ky\, e^{-ky}dy \qquad (2.7)$$

(Whittaker 1918), provided that the 'ratio of the smallest root to every one of the others is small'.

Suppose now that $k \to 0$. Then

$$c_{p0}^2 \approx \int_0^\infty N^2(y)y\,dy - k\int_0^\infty N^2(y)y^2dy+\dots, \qquad (2.8)$$

where c_{p0} is the long-wave phase speed of a mode 1 wave.

2.2. *Large k*

The angular frequency of an internal wave must be less than the maximum buoyancy frequency, for otherwise equation (1.1) would not have oscillatory solutions. Hence $\omega \leqslant N_{\max}$. Provided that $N^2(y)$ is bounded it is possible to prove by means of Stürm's comparison theorems (Yih 1965, pp. 31–33) that the angular frequency of an internal wave must be an increasing function of k. Note that when $N^2(y)$ is unbounded, classical Stürm–Liouville theory (including the Stürm comparison theorems) is not valid. For example, when $N^2(y)$ consists of one or more delta functions (corresponding to step discontinuities in the density function) there will only be a finite set of modes. Hence, when the stratification is continuous, and $k \to \infty$, then $\omega \to \omega_{\max}$ and $1/c_p \sim k/\omega_{\max}$ and so in equation (1.3), $c/l \sim \omega_{\max}$, where $\omega_{\max} \leqslant N_{\max}$. Hence a result such as (1.3) might be expected for large k in the case of continuous stratification.

On the other hand, if $k \to \infty$ in equation (2.7) then

$$c_p^2 \approx \frac{1}{k}\int_0^\infty \frac{N^2(y)}{2}dy \qquad (2.9)$$

and so $1/c_p = O(k^{\frac{1}{2}})$ for large k. This is at variance with the result presented in the previous paragraph, an apparent contradiction which is resolved by the discussion of the two-layer case which follows next.

2.3. *The two-layer case*

It is instructive to compare the foregoing results with the case of two layers, the depth of the interface being d and the layer below the interface being of infinite depth. In this case

$$c_p^2 = \frac{g}{k}\frac{\Delta\rho}{\rho_0}\frac{1}{1+\coth kd} \qquad (2.10)$$

(Phillips 1977, p. 213). $\Delta\rho$ is the difference in density between the two layers and ρ_0 is one half of the sum of the densities of the two layers. It follows that as $kd \to 0$ then $c_p^2 \sim gd\Delta\rho/\rho_0$. This result also follows directly from equation (2.7) on substituting a delta function for $N^2(x)$. Hence (2.7) gives the exact result in the case of two layers. This is not surprising since all modes above the first disappear in the case of two layers and if a delta function is substituted for $N^2(x)$ into the iterated solution (2.5) of the integral equation (2.2) it will be found that all the iterations above the first vanish. Consequently (2.7) must then give the exact result. Hence (2.8) may be expected to give a reasonable result generally when the water column can be approximated by two layers. Actually, it will be demonstrated in §§5 and 6 that the approximation is reasonable even when the water column is totally unlike two layers.

On the other hand, as $kd \to \infty$ then $c_p^2 \sim g\Delta\rho/2k\rho_0$ and so, when $N^2_{\max}$ is infinite, $1/c_p = O(k^{\frac{1}{2}})$. Since equation (2.7) is essentially a two-layer approximation it may be expected to give the same result, and this was shown to be the case in §2.2.

2.4. *General conclusions*

In summary, the above arguments show that as $k \to \infty$, $1/c_p \to k/\omega_{\max}$ when $N^2_{\max}$ is finite and the stratification is continuous, and $1/c_p = O(k^{\frac{1}{2}})$ when $N^2_{\max}$ is infinite and the density function is therefore not continuous. When $k \to 0$, $1/c_p \to 1/c_{p0}$ where c_{p0} is given approximately by (2.8), a result which is exact for two layers. For a continuous density function the dispersion relation can therefore be expressed as

$$\frac{1}{c_p} = \frac{1}{c_{p0}} + \frac{k}{\omega_{\max}} + \varepsilon(k), \tag{2.11}$$

where $\varepsilon(0) = 0$ and $\varepsilon(k) = o(k)$ for large k. In addition, $\varepsilon(k)$ must be bounded for finite k since, as explained above in §2.2, ω is an increasing function of k and $1/c_p$ is therefore finite for finite k. The approximate dispersion relation (1.3) is therefore valid if $\varepsilon(k)$ is small in some sense, which has so far been shown to be the case for small enough k and large enough k. If $\varepsilon(k)$ can be neglected the approximate dispersion relation for a mode 1 wave is

$$\frac{1}{c_p} \approx \left(\int_0^\infty N^2(y)y\,\mathrm{d}y\right)^{-\frac{1}{2}} + \frac{k}{\omega_{\max}}. \tag{2.12}$$

In the examples to be discussed later, $\omega_{\max} = N_{\max}$.

It remains now to analyse the problem further by comparing equations (1.3) and (2.11) with more precise results from specific examples. We start first with the choice of a suitable density function.

3. The density function

Take the density function to be

$$\rho(x) = \rho_0 + \Delta\rho\, f(x), \tag{3.1}$$

where $f(x)$ is a continuous real-valued function defined on $[0,\infty]$. It is assumed throughout this paper that the water depth is infinite. Let $f(0) = 0$ (so that $\rho(0) = \rho_0 > 0$), and let $f(x) \to 1$ as $x \to \infty$ (so that $\rho(\infty) = \rho_0 + \Delta\rho$). For example $f(x) = 1 - \mathrm{e}^{-\beta x}$ satisfies these requirements. Moreover this function appears to be a rough approximation to the measured density function in figure 3.

3.1. *The exponential density function*

In this analysis x is taken to be real, where $x \in [0,\infty]$. The interval $[0,\infty]$ is mapped on to $[1,0]$ by the transformation $\chi = \mathrm{e}^{-\sigma x}$ and so substituting χ for $\mathrm{e}^{-\sigma x}$ gives $f(\chi)$ where $\chi \in [0,1]$. The Weierstrass approximation theorem (Cheney 1966, p. 65 *et seq*) implies that a set of linear combinations of χ^r is dense on $[0,1]$, spans this interval, and asserts that a unique polynomial exists which approximates $f(\chi)$ uniformly on $[0,1]$ as n increases. This polynomial is here identified with $\sum_{r=0}^{n} \alpha_r \chi^r$. As $n \to \infty$, it approximates $f(\chi)$ exactly on the closed interval $[0,1]$. Transforming back to $x \in [0,\infty]$ then gives the result that $f(x)$ can be approximated uniformly on $[0,\infty]$ by

$$f(x) = \sum_{r=0}^{n} \alpha_r \mathrm{e}^{-r\sigma x}, \qquad x \in [0,\infty] \tag{3.2}$$

and in addition, in order to satisfy the requirements of (3.1) we must have $\alpha_0 = 1$ and $\sum_{r=0}^{n} \alpha_r = 0$.

Equation (3.2) gives $f(x) - 1 = O(\mathrm{e}^{-\sigma x})$ as $x \to \infty$, but $f(x) - 1$ is $O(\mathrm{e}^{-\beta x})$ and so $\sigma \equiv \beta$. The parameter β is thus unique, and the exponential expansion

$$f(x) = \sum_{r=0}^{n} \alpha_r \mathrm{e}^{-r\beta x}, \qquad x \in [0,\infty] \tag{3.3}$$

is also unique on $[0,\infty]$.

This paper is concerned with a theoretical investigation of the dispersion relation and the numerical modelling of experimental data is beyond this scope. However, some comments on the expansion (3.3) are appropriate. In practice it may be quite difficult to determine β with any precision. The reason is that $f(x)$ is determined from experimental data measured over a finite interval. The requirement that $f(x)-1$ be $O(\mathrm{e}^{-\beta x})$ at infinity may thus not be easy to enforce and when this constraint is removed β may not be unique. Only the case of finite n is considered in this paper. If n is finite then $f(x)$ may be approximated 'in the mean' by minimizing a norm such as in a 'least squares' approximation. This can be accomplished by choosing a value for β and then computing a set $\{\alpha_i\}$ by least squares, thus minimizing a square norm, which however, depends on β. Hence the norm must be minimized 'globally' by iterating on β, computing new sets $\{\alpha_i\}$ until it is mininized with respect to $\{\alpha_i, \beta\}$. This approximation will be unique (given n), but as the order of the polynomial, n, is increased, the minimum of the norm will rapidly become less well defined and eventually almost independent of the parameter β. Any practical algorithm will then fail because of the finite numerical precision of computers. Hence n should be restricted to a fairly small value.

There a number of additional aspects which should be mentioned in this section. For example, when n is finite, as it always must be in any practical application, it is probable that although the norm has a distinct global minimum there are many local minima. In any practical situation the density function is fitted to a finite set of points rather than a continuous function. This is another situation where there may be many local minima. These local minima are certainly found in practice and cause some difficulties. Again, this is a good reason to limit n to a small value.

Finally, note that a function of the form of (3.3) implies that $\rho(x)$ and all its derivatives are continuous functions of x. This function cannot therefore be used to represent the case of a water column which has a density structure consisting of a sequence of steps, unless the steps have finite derivatives (which they do have in reality).

3.2. *The incomplete beta function density*

Included in (3.3) is an interesting special case which will used as an example in later sections of this paper. Consider

$$f(x) = (p+1)\beta \int_0^x \mathrm{e}^{-\beta t}(1-\mathrm{e}^{-\beta t})^p \mathrm{d}t. \tag{3.4}$$

Notice that $f(x)$ is a monotonically increasing function of x. Also $f(0) = 0$, and $f(\infty) = 1$. This function is an incomplete beta function (Erdélyi *et al.* 1953, p. 87). It represents a step, the depth and width of which are functions of p and β.

3.3. *The buoyancy frequency*

From equations (1.2), (3.1) and (3.3)

$$N^2(x) = -g\beta \frac{\Delta\rho}{\rho_0} \sum_{r=1}^{n} r\alpha_r \mathrm{e}^{-r\beta x} \quad + O(\Delta\rho/\rho_0)^2. \tag{3.5}$$

Note that g is negative since positive x is downwards and hence, if $\eta_r^2 = -g\beta r\alpha_r \Delta\rho/\rho_0$ then

$$N^2(x) = \sum_{r=1}^{n} \eta_r^2 \mathrm{e}^{-r\beta x} \quad + O(\Delta\rho/\rho)^2. \tag{3.6}$$

In the case of the beta density (3.4)

$$N^2(x) = -(p+1)\beta g \frac{\Delta\rho}{\rho_0} \mathrm{e}^{-\beta x}(1-\mathrm{e}^{-\beta x})^p \quad + O(\Delta\rho/\rho)^2. \tag{3.7}$$

And thus if $\gamma^2 = -\beta g \Delta\rho/\rho_0$, and terms $O(\Delta\rho/\rho)^2$ are ignored from now on, then

$$N^2(x) = \gamma^2(p+1)\mathrm{e}^{-\beta x}(1-\mathrm{e}^{-\beta x})^p. \tag{3.8}$$

It is straightforward to show that $\mathrm{d}^m\left[N^2(x)\right]/\mathrm{d}x^m = 0$ at $x = 0$ for all $m < p$ including $m = 0$. So when p is large, N^2 increases slowly at first from 0 at $x = 0$, and then more rapidly to a peak, and then decreases exponentially towards zero at $x = \infty$. When $p = 1$, N^2 increases rapidly at first from 0 at $x = 0$, and then increases more slowly to a peak at $x = \ln 2/\beta$, and then decreases exponentially towards zero at $x = \infty$.

The position of the peak is found from $d\left[N^2(x)\right]/dx = 0$

$$x_{\text{peak}} = \frac{1}{\beta}\ln(p+1). \tag{3.9}$$

The separation of the points at which $\mathrm{d}^2N^2(x)/\mathrm{d}x^2 = 0$ gives a definition of the width of the peak, and is

$$\Delta x = \frac{1}{\beta}\ln\left(\frac{3p+2+[p(5p+4)]^{\frac{1}{2}}}{3p+2-[p(5p+4)]^{\frac{1}{2}}}\right). \tag{3.10}$$

When $p = 1$ there is only one point at which $\mathrm{d}^2N^2(x)/\mathrm{d}x^2 = 0$ ($x = \ln 4/\beta$) and the width is then defined as the distance between this point and $x = 0$. This case is then also included in the above formula.

So for $p = 1$, $\Delta x = \ln 4/\beta = 1.3863/\beta$; for $p = 10$, $\Delta x = 1.8412/\beta$, and for $p = 100$, $\Delta x = 1.9160/\beta$. As $p \to \infty$, $\Delta x \to 1.9248/\beta$.

4. The differential equation

If all terms $O(\Delta\rho/\rho_0)^2$ are ignored, and $N^2(x)$ is represented by (3.6), the differential equation (1.1) becomes

$$\psi''(x) + \left(\frac{1}{c_p^2} \sum_{r=1}^{n} \eta_r^2 e^{-r\beta x} - k^2 \right) \psi(x) = 0. \tag{4.1}$$

Put $t = e^{-\beta x}$ and then

$$t^2\psi'' + t\psi' + \left(\frac{1}{\beta^2 c_p^2} \sum_{r=1}^{n} \eta_r^2 t^r - \frac{k^2}{\beta^2} \right) \psi = 0. \tag{4.2}$$

This equation has a regular singularity at $t = 0$ and an irregular singularity at $t = \infty$. When $n = 1$ or $n = 2$ the equation is a confluent hypergeometric equation (the case $n = 1$ is transformed into a special case of $n = 2$ by the transformation $t = v^2$). For $n > 2$ this is no longer so (Ince 1926, §20.51). For $n > 2$ therefore, the equation does not have known transcendental solutions. However, one may hope that in this case it is possible to express the solutions to the equation approximately in terms of the solution for $n = 2$ using a method of comparing solutions. In the next section equation (4.2) is solved for the case when $n = 1$ and this leads to Bessel's equation. In the subsequent section the case $n = 2$ is shown to lead to Kummer's equation. In both cases the approximate dispersion relation (1.3) is shown to be valid both for the lowest mode and also for higher modes.

5. The case $n=1$: Bessel's equation

The Bessel function case has been analysed before (Garrett & Munk 1972). Here, it is shown how the eigenvalues relate to the dispersion relation (1.3).

When $n = 1$ equation (4.2) becomes

$$t^2\psi'' + t\psi' + \left(\frac{\eta_1^2 t}{\beta^2 c_p^2} - \frac{k^2}{\beta^2} \right) \psi = 0. \tag{5.1}$$

Putting $t = w^2(\beta^2 c_p^2/4\eta_1^2)$ then gives

$$w^2\psi'' + w\psi' + \left(w^2 - \frac{4k^2}{\beta^2} \right) \psi = 0 \tag{5.2}$$

which is Bessel's equation, independent solutions of which are $J_{\frac{2k}{\beta}}(w)$ and $Y_{\frac{2k}{\beta}}(w)$. In terms of the original parameters the solution is

$$\psi = C\, J_{\frac{2k}{\beta}} \left(\frac{2\eta_1}{\beta c_p} e^{-\frac{\beta x}{2}} \right) + D\, Y_{\frac{2k}{\beta}} \left(\frac{2\eta_1}{\beta c_p} e^{-\frac{\beta x}{2}} \right), \tag{5.3}$$

where C and D are arbitrary constants. For infinitely deep water one boundary condition is $\psi \to 0$ as $x \to \infty$ and hence

$$\psi = C\, J_{\frac{2k}{\beta}} \left(\frac{2\eta_1}{\beta c_p} e^{-\frac{\beta x}{2}} \right). \tag{5.4}$$

The other boundary condition is $\psi = 0$ at $x = 0$ and so

$$J_{\frac{2k}{\beta}} \left(\frac{2\eta_1}{\beta c_p} \right) = 0. \tag{5.5}$$

The eigenvalues are therefore zeros of the ordinary Bessel function, and the argument of the Bessel function is proportional to $1/c_p$. A plot of the zeros of $J_\nu(x)$ (Watson 1944, figure 33) shows that for $\nu \geqslant 0$ and $x \geqslant 0$ there is an almost linear relation between the zero, $j_{\nu,n}$ and ν. The smallest zero of $J_\nu(x)$ for large ν is given by $j_{\nu,1} = \nu + O(\nu^{\frac{1}{3}})$ (Watson 1944, §18.81). Here, $\nu = 2k/\beta$, and so, for large ν, $1/c_p \sim k/\eta_1$, or, since $N_{\max} = \eta_1$, then $1/c_p \sim k/N_{\max}$, and in this case $\omega_{\max} = N_{\max}$, see §2.4.

An approximation for the mth zeros of $J_\nu(x)$ is

$$j_{\nu,m} \approx \left(m + \frac{\nu}{2} - \frac{1}{4}\right)\pi - \frac{4\nu^2 - 1}{8\pi\left(m + \frac{\nu}{2} - \frac{1}{4}\right)} - \dots \tag{5.6}$$

(Watson 1944, §15.53). This approximation is good for moderate $\nu \leqslant 1$ if m is small. Hence

$$\frac{1}{c_p} \approx \frac{\pi k}{2\eta_1} + \frac{\pi\beta}{2\eta_1}\left(m - \frac{1}{4}\right). \tag{5.7}$$

Comparing this equation with equation (1.3) it is clear that in this case, at least, there is qualitative agreement with the computations based on experimental data. When $k = 0$ and $m = 1$, formula (5.7) gives $c_{p0} \approx 8\eta_1/3\pi\beta$, whereas formula (2.8) gives $c_{p0} \approx \eta_1/\beta$, a difference of approximately 15%.

6. The case n=2: Kummer's equation

This case is of particular interest because if the second term in (3.6) is negative and $\eta_1^2 = \zeta_1^2$ and $\eta_2^2 = -\zeta_2^2$ so that $N^2(x) = \zeta_1^2 e^{-\beta x} - \zeta_2^2 e^{-2\beta x}$ then $N^2(x)$ has a smooth maximum and provides a rough model for the observed frequency function.

In this case the equation becomes

$$t^2\psi'' + t\psi' + \left(\frac{1}{\beta^2 c_p^2}(\zeta_1^2 t - \zeta_2^2 t^2) - \frac{k^2}{\beta^2}\right) = 0. \tag{6.1}$$

Put $t = u(\beta c_p/2\zeta_2)$, $\mu = \zeta_1^2/2c_p\beta\zeta_2$, and $\xi = k/\beta$ in equation (6.1), and then

$$u^2\psi'' + u\psi' + \left(\mu u - \frac{u^2}{4} - \xi^2\right)\psi = 0. \tag{6.2}$$

There are two possible values of ξ which satisfy (6.2) ($+\xi$ and $-\xi$). The positive exponent is chosen since it gives that branch of the solution which is zero at $u = 0$. Hence put $\psi(u) = u^\xi e^{-u/2}\Psi(u)$, and then

$$u\Psi'' + (1 + 2\xi - u)\Psi' - (\tfrac{1}{2} + \xi - \mu)\Psi = 0. \tag{6.3}$$

This is Kummer's form of the confluent hypergeometric equation (Slater 1960, §1.1.1). In terms of Kummer's function, that branch which satisfies the boundary condition $\Psi(0) = 0$ is

$$\Psi = E\ {}_1F_1(\tfrac{1}{2} + \xi - \mu; 1 + 2\xi; u), \tag{6.4}$$

where E is an arbitrary constant. In terms of the original parameters this is

$$\psi = E\left(\frac{2\zeta_2}{c_p\beta}\right)^{k/\beta} \exp\left(-kx - \frac{\zeta_2}{c_p\beta}e^{-\beta x}\right)\ {}_1F_1\left(\frac{1}{2} + \frac{k}{\beta} - \frac{\zeta_1^2}{2c_p\beta\zeta_2}; 1 + \frac{2k}{\beta}; \frac{2\zeta_2}{c_p\beta}e^{-\beta x}\right). \tag{6.5}$$

The second boundary condition requires that at $x = 0$, $\psi = 0$ and therefore the eigenvalues are given by

$$ {}_1F_1\left(\frac{1}{2}+\frac{k}{\beta}-\frac{\zeta_1^2}{2c_p\beta\zeta_2};1+\frac{2k}{\beta};\frac{2\zeta_2}{c_p\beta}\right)=0. \tag{6.6}$$

Let $\tau = 2\zeta_2/c_p\beta$, also $\zeta_2^2 > 0$ by hypothesis, hence the zero, τ, is real. In addition, put $a = \frac{1}{2} + k/\beta - \zeta_1^2/2c_p\beta\zeta_2$, and $b = 1 + 2k/\beta$ (hence $b > 0$). If $-1 < a < 0$, $b > 0$ and $u > 0$ there will be one real zero; similarly if $-2 < a < -1$ there will be two real zeros and so on (Slater 1960, §6.1). One zero coincides with the surface and therefore, for the mth mode

$$\tau > \frac{2\zeta_2^2}{\zeta_1^2}\left(\frac{2k}{\beta}+(2m-1)\right) \tag{6.7}$$

or, since $\tau = 2\zeta_2/\beta c_p$

$$\frac{1}{c_p} > \frac{2k\zeta_2}{\zeta_1^2}+\frac{\beta\zeta_1}{\zeta_1^2}(2m-1). \tag{6.8}$$

Notice that, because $-m < a < -(m-1)$ for m zeros, then, for a given m, a is bounded as $k \to \infty$. Now, $a = \frac{1}{2} + k/\beta - \zeta_1^2/2c_p\beta\zeta_2$, and therefore, as $k \to \infty$, $\zeta_1^2/2c_p\beta\zeta_2 \sim k/\beta$, *i.e.* $1/c_p \sim 2\zeta_2 k/\zeta_1^2$. It is straightforward to show that $N_{\max} = \eta_1^2/2\zeta_2$ and hence $1/c_p \sim k/N_{\max}$ for all modes. Compare this with the remarks made in §2.4; this is an example where $\omega_{\max} = N_{\max}$.

It is possible to estimate the zeros of the Kummer function in a number of ways. Because a is bounded as $k \to \infty$ for a given mode number m the following result may be used here:

$${}_1F_1[a;b;u]=\frac{e^{u/2}\Gamma(b)}{(2\mu)^{b-\frac{2}{3}}}[\mathrm{Ai}(s)\cos a\pi+\mathrm{Bi}(s)\sin a\pi+O(\mu^{-\frac{2}{3}})], \tag{6.9}$$

where $s = (u/4\mu - 1)(2\mu)^{\frac{2}{3}}$ and $\mu = b/2 - a$ (Tricomi 1954, p. 123); (Slater 1960, equation 4.5.7). Ai, Bi are the usual Airy functions, and at the zero, $u = 2\zeta_2/c_p\beta$. Also, $\mu = \zeta_1^2/2c_p\beta\zeta_2$. This approximation is valid when u and 4μ are approximately equal and go to ∞ together so that Ai(s) is 'large' in comparison with $\mu^{-\frac{2}{3}}$. This approximation then gives

$$\mu=\xi+\frac{1}{2}+(m-1)+\frac{1}{\pi}\tan^{-1}\left(\frac{\mathrm{Ai}(s)+O(\mu^{-\frac{2}{3}})}{\mathrm{Bi}(s)}\right), \tag{6.10}$$

where m is the mode number. Now, $u/4\mu = \zeta_2^2/\zeta_1^2$, $s = \mu^{\frac{2}{3}}(\zeta_2^2/\zeta_1^2 - 1)$. Hence

$$\frac{1}{c_p}=\frac{2k\zeta_2}{\zeta_1^2}+\frac{\beta\zeta_2}{\zeta_1^2}\left((2m-1)+\frac{2}{\pi}\tan^{-1}\left\{\frac{\mathrm{Ai}[\mu^{\frac{2}{3}}(\zeta_2^2/\zeta_1^2-1)]+O(\mu^{-\frac{2}{3}})}{\mathrm{Bi}[\mu^{\frac{2}{3}}(\zeta_2^2/\zeta_1^2-1)]}\right\}\right). \tag{6.11}$$

Compare with equation (2.11). Note that $N_{\max} = \zeta_1^2/2\zeta_2$, and so $1/c_p \sim 2k\zeta_2/\zeta_1^2$, *i.e.* $\sim k/N_{\max}$ for large k.

When $\zeta_1 = \zeta_2 = \zeta$, $N^2(0) = 0$ in the x domain, and then $u = 4\mu$ and $s = 0$. Furthermore $\mathrm{Ai}(0)/\mathrm{Bi}(0) = 1/\sqrt{3}$ and so

$$\frac{1}{c_p}=\frac{2k}{\zeta}+\frac{\beta}{\zeta}\left((2m-1)+\frac{1}{3}+O(\mu^{-\frac{2}{3}})\right). \tag{6.12}$$

For $k = 0$ and $m = 1$ this formula gives $c_{p0} \approx 3\zeta/4\beta$ whereas formula (2.8) gives $c_{p0} \approx \sqrt{3}\zeta/2\beta$, a difference of about 14%.

7. The general case

In the previous two sections two special cases of the $N^2(t)$ function corresponding to $n = 1$ and $n = 2$ were examined. The eigenfunctions were found to be Bessel functions and confluent hypergeometric functions respectively. It was shown that, in each case, $\omega_{\max} = N_{\max}$ for all modes. This relation is now shown to be true for mode 1 for the 'beta density' of §3.2 and a sufficient condition for more general validity is derived for mode 1. In particular, the following result is proved in this section: If $N^2(x)$ is $O(e^{-\beta x})$ as $x \to \infty$ and has a maximum value $N^2_{\max}$ then a sufficient condition for

$$\frac{1}{c_p} \sim \frac{k}{N_{\max}} \tag{7.1}$$

to hold for large k for the lowest mode is that $N^2(t)/t$ is convex for $0 \leqslant t \leqslant 1$ where $t = \mathrm{e}^{-\beta x}$, and $1/\beta$ is an arbitrary length scale.

7.1. *The asymptotic approximation*

In the general case the differential equation (4.2) gives

$$t^2\psi'' + t\psi' + \left(\frac{N^2(t)}{\beta^2 c_p^2} - \frac{k^2}{\beta^2}\right)\psi = 0. \tag{7.2}$$

In the analysis which follows it will be more convenient to remove the term in ψ' in equation (7.2) by means of the substitution $\psi = t^{-\frac{1}{2}}\phi$ to give

$$\phi'' + \left(\frac{1}{\beta^2 c_p^2}\frac{N^2(t)}{t^2} - \frac{(\xi^2 - \frac{1}{4})}{t^2}\right)\phi = 0, \tag{7.3}$$

where $\xi = k/\beta$. When a power series such as equation (3.6) is substituted for $N^2(t)$ with $t = \mathrm{e}^{-\beta x}$ it is clear that the resulting equation is closely related to Whittaker's equation in that it has a regular singularity at $t = 0$ and an irregular singularity at $t = \infty$ (Slater 1960, §1.6). In general, however, the equation has an irregular singularity at ∞ of higher rank than Whittaker's equation and few properties of the solutions have been investigated. The confluent hypergeometric case was worked out in the previous section, and it is clearly of some interest to find out if equation (7.3) can be approximated in some sense by Whittaker's equation. In this section it is proved that this is indeed the case when $k \to \infty$. In particular, it was shown in §2 that as $k \to \infty$, $1/c_p \sim k/\omega_{\max}$ where $\omega_{\max} \leqslant N_{\max}$. In §§5 and 6 it was demonstrated that for two specific cases of $N^2(t)$, $\omega_{\max} = N_{\max}$. It is now shown that this relation is more generally true.

Consider the 'turning points' where the angular frequency of the wave ω is equal to the local buoyancy frequency $N(t)$. If $N^2(t)$ has a single maximum there will be two turning points. The domain between the two turning points contains the oscillating part of the eigenfunction where there are maxima, minima, and zeros. The eigenfunction decays exponentially in the x domain on either side of the turning points. If k is large, then the exponential decay will be very rapid. Hence, for large enough k, it might be expected that the eigenfunction will be significantly different from zero (in some sense) only in the t and x domains between the turning points. Consequently, the behaviour of the eigenfunction (and the eigenvalues) for large enough k might be largely determined by the $N^2(t)$ function in the neighbourhood of the maximum.

In order to keep the following analysis manageable it will be assumed that $N^2(t)$ has one maximum. In a sufficiently small neighbourhood of the maximum, $N^2(t)$

has a parabolic form. In the previous section it was shown that when $N^2(t)$ is a quadratic the eigenfunction is a confluent hypergeometric function, and for this case, $1/c_p \sim k/N_{\max}$. Notice that the relation $1/c_p \sim k/N_{\max}$, if generally true, is independent of the exact shape of the $N^2(t)$ function. It is only dependent on the maximum value. If $N^2(t)$ is approximated by a parabola it ought not to be necessary to approximate it by an 'exact' Taylor series parabolic approximation near the peak. Any parabola ought to do, so long as it has the required peak value.

Now consider the general buoyancy squared frequency function $N^2(t)$. Let $N^2(t)$ be approximated by $M^2(t)$, a function to be determined. In addition to equation (7.3) we then have

$$\theta'' + \left(\frac{1}{\beta^2 c_\theta^2}\frac{M^2(t)}{t^2} - \frac{(\xi^2 - \frac{1}{4})}{t^2}\right)\theta = 0, \tag{7.4}$$

where θ is the eigenfunction associated with $M^2(t)$. The eigenvalues associated with $N^2(t)$ and $M^2(t)$ are $1/c_\phi$ and $1/c_\theta$ respectively.
Also

$$(\phi'\theta - \phi\theta')' = \phi''\theta - \phi\theta'', \tag{7.5}$$

where the prime denotes $\mathrm{d}/\mathrm{d}t$. Hence

$$\frac{1}{\beta^2}\int_0^1 \frac{M^2(t)}{c_\phi^2 t^2}\phi\theta \mathrm{d}t - \frac{1}{\beta^2}\int_0^1 \frac{N^2(t)}{c_\theta^2 t^2}\phi\theta \mathrm{d}t = \left[\phi'\theta - \phi\theta'\right]_0^1. \tag{7.6}$$

Suppose that ϕ and θ satisfy the boundary conditions $\phi(0) = \theta(0) = \phi(1) = \theta(1) = 0$ and that ϕ' and θ' are bounded at $t = 0$ and $t = 1$ so that ϕ and θ are eigenfunctions. Then

$$c_\phi^2/c_\theta^2 = \int_0^1 \frac{N^2}{t^2}\phi\theta \mathrm{d}t \Big/ \int_0^1 \frac{M^2}{t^2}\phi\theta \mathrm{d}t. \tag{7.7}$$

The integrals in expressions (7.6) and (7.7) converge if $\xi > 0$ because near $t = 0$, $\phi_i \sim t^\xi$, $\theta_i \sim t^\xi$, $M^2 = O(t)$ and $N^2 = O(t)$.

Consider the lowest mode. Then $\phi\theta \geqslant 0$ and thus $N^2\phi\theta/t^2 \geqslant 0$ and $M^2\phi\theta/t^2 \geqslant 0$. Suppose that it is possible to choose $M^2(t)$ to be a quadratic such that $M^2(t) \leqslant N^2(t)$ and $M^2(0) = 0$. Consequently, $c_\phi \geqslant c_\theta$. Furthermore, let the maximum values of $M^2(t)$ and $N^2(t)$ be equal so that $M^2_{\max} = N^2_{\max}$. Since $M^2(t)$ is a quadratic the eigenfunction $\theta(t)$ is a confluent hypergeometric function and it was shown in §6 that for this case $c_\theta \sim N_{\max}/k$. Also, $c_\phi \sim \omega_{\max}/k$ where $\omega_{\max} \leqslant N_{\max}$, but $c_\phi \geqslant c_\theta$ hence $\omega_{\max} = N_{\max}$ and hence

$$\frac{1}{c_\phi} \sim \frac{k}{N_{\max}} \tag{7.8}$$

for mode 1, provided that $M^2(t)$ and $N^2(t)$ satisfy the conditions specified above. These conditions are now examined in detail.

Equation (3.6) with $t = \mathrm{e}^{-\beta x}$ gives a polynomial in t for $N^2(t)$

$$N^2(t) = \sum_{r=1}^{n} \eta_r^2 t^r. \tag{7.9}$$

Let $N^2(t)$ have one maximum in $0 \leqslant t \leqslant 1$ at $t = t_0$, at which point $N^2(t_0) = N^2_{\max}$. Suppose that $N^2(t)$ is approximated by $M^2(t)$ where

$$M^2(t) = N^2_{\max}\frac{t}{t_0}\left(2 - \frac{t}{t_0}\right) \tag{7.10}$$

so that $M^2(t_0) = N^2_{max}$ and $M^2(0) = 0$. Notice that, as defined in (7.10), $M^2(t)$ will be negative when $t > 2t_0$. This is not physically realistic since such a water column would be unstable. However, this does not matter since as $k \to \infty$, the eigenfunction is arbitrarily small in the region where $t > 2t_0$ compared with its magnitude where $t = t_0$. Hence $M^2(t)$ could be defined as zero where $t > 2t_0$ without affecting the result. To do so, however, would unnessarily complicate the analysis.

Now define

$$R^2(t) = N^2(t) - M^2(t). \tag{7.11}$$

Clearly, since $M^2(t_0) = N^2(t_0)$ then $R^2(t_0) = 0$. Now consider

$$\frac{R^2(t)}{t} = \sum_{r=1}^{n} \eta_r^2 t^{r-1} - \frac{N^2_{max}}{t_0}\left(2 - \frac{t}{t_0}\right). \tag{7.12}$$

Since $R^2(t_0) = 0$, $M^2(t)/t$ is a linear function which intersects $N^2(t)/t$ at t_0, the maximum of $N^2(t)$. Furthermore, if $N^2(t)/t$ is a convex function then $M^2(t)/t$ will be a tangent to $N^2(t)/t$ at t_0 and consequently $M^2(t)/t \leqslant N^2(t)/t$ (Hardy, Littlewood & Pólya 1959, theorem 112). Hence it is sufficient that $N^2(t)/t$ is a convex function for the condition $M^2(t) \leqslant N^2(t)$ to hold, as required in the proof of (7.8).

As an example take the 'beta density' $N^2(t) = \gamma^2(p+1)t(1-t)^p$ described in §3.2. The maximum occurs at $t_0 = 1/(p+1)$ at which point $N^2_{max} = \gamma^2 p^p/(p+1)^p$. It is straightforward to show that $\mathrm{d}^2[N^2(t)/t]/\mathrm{d}t^2 \geqslant 0$, and hence $N^2(t)/t$ is a convex function (Hardy, Littlewood & Pólya 1959, theorem 94). Expression (7.8) thus holds for the 'beta density' for the lowest mode.

8. Multiple peaks: a qualitative description

Experimentally measured buoyancy frequency functions often consist of a basic 'hump' with multiple peaks and troughs superimposed. The methods of the previous section can be used to provide a qualitative description of the effect of multiple peaks in the $N^2(x)$ function on the dispersion relation for the lowest mode. As an illustration consider the beta density, equation (3.8), modified by the addition of oscillations

$$Q^2(t) = \gamma^2(p+1)t(1-t)^p[1+h(t)], \tag{8.1}$$

where $t = \mathrm{e}^{-\beta x}$, as usual. Also, $h(0) = 0$, $h(1) = 0$, and $h(t) \geqslant -1$ since $Q^2(t) \geqslant 0$. In addition, $h(t)$ has maxima and minima so that it oscillates about 0 in $0 \leqslant t \leqslant 1$. Hence $Q^2(t)$ is basically the beta density of §3.2 with oscillations superimposed. Let

$$S^2(t) = \gamma^2(p+1)t(1-t)^p \tag{8.2}$$

so that $Q^2(t) = S^2(t)[1+h(t)]$, and let $\Phi(t)$ and $\Theta(t)$ be the eigenfunctions corresponding to $Q^2(t)$ and $S^2(t)$ respectively. Furthermore let the corresponding eigenvalues be $1/c_\Phi$ and $1/c_\Theta$. Applying equation (7.7) then gives

$$c^2_\Phi/c^2_\Theta = \int_0^1 \frac{Q^2}{t^2}\Phi\Theta\,\mathrm{d}t \bigg/ \int_0^1 \frac{S^2}{t^2}\Phi\Theta\,dt = 1 + \int_0^1 \frac{S^2 h}{t^2}\Phi\Theta\,dt \bigg/ \int_0^1 \frac{S^2}{t^2}\Phi\Theta\,dt. \tag{8.3}$$

It is evident that if $h(t)$ oscillates over $0 \leqslant t \leqslant 1$ and does so rapidly enough then the integral $\int_0^1 (S^2/t^2)h\Phi\Theta\,\mathrm{d}t$ will be small in some sense. In particular, suppose that the scale length of each oscillation is much less than the scale length of the main hump, *i.e.* $\ll 1/\beta$ and also much less than the scale length of the eigenfunction. Then it may be expected that the integral will be small and that $c_\Phi \approx c_\Theta$. Hence the effective buoyancy frequency squared function is a smoothed version of the actual function if k is small enough and the dispersion relation will be given by (2.11) with $\omega_{\max}$ set by the peak of the underlying smooth function (8.2). As k increases the peaks of the oscillations will become signficant and the effective $\omega_{\max}$ will increase so that the gradient of the term in (2.11) proportional to k will decrease.

9. Summary and discussion

In the Introduction it was noted that the dispersion relation for linearly propagating internal waves, computed numerically from the eigenfunction equation using experimental data for the density function, often has a particularly simple form. This simple approximate dispersion relation is especially useful for predicting the general shape and pattern of internal wave wakes.

In §2 some general properties of the dispersion relation were examined and it was demonstrated that in general

$$\frac{1}{c_p} = \frac{1}{c_{p0}} + \frac{k}{\omega_{\max}} + \varepsilon(k)$$

where c_p is the wave phase speed for a particular mode, c_{p0} is the phase speed at $k = 0$, $\omega_{\max}$ is the maximum possible wave angular frequency and $\omega_{\max} \leqslant N_{\max}$ where $N_{\max}$ is the maximum buoyancy frequency. Also, $\varepsilon(k) = 0$ at $k = 0$, is bounded for finite k and $\varepsilon(k) = o(k)$ when k is large.

In particular, when $\varepsilon(k)$ can be neglected, the dispersion relation for a mode 1 wave is approximately

$$\frac{1}{c_p} \approx \left(\int_0^{\infty} N^2(y)y\,\mathrm{d}y\right)^{-\frac{1}{2}} + \frac{k}{\omega_{\max}}.$$

Interest then centres on finding sufficient conditions for $\omega_{\max} = N_{\max}$, and in discovering if $\varepsilon(k)$ can be neglected. These problems are addressed by analysing the eigenvalue problem for a class of buoyancy frequency squared functions $N^2(x)$ which is taken to be a class of real-valued functions of a real variable x where $0 \leqslant x \leqslant \infty$ such that $N^2(x) = O(\mathrm{e}^{-\beta x})$ as $x \to \infty$ and $1/\beta$ is an arbitrary length scale. Only the infinite-depth problem with a rigid surface is considered. It was demonstrated that $N^2(x)$ can be represented by a power series expansion in $\mathrm{e}^{-\beta x}$. The eigenfunction equation was constructed for such a function and two cases of the equation which have solutions in terms of known functions (Bessel functions and confluent hypergeometric functions) were worked out. More generally, it was demonstrated that when $k \to \infty$ it is possible to approximate the equation uniformly in such a way that it can be compared with the confluent hypergeometric equation. The eigenfunctions of the equation can then be taken to be Whittaker functions as an approximation and the eigenvalues are, approximately, zeros of the Whittaker functions. The main result which follows from this approach is the following:

If $N^2(x)$ is $O(\mathrm{e}^{-\beta x})$ as $x \to \infty$ and has a maximum value $N^2_{\max}$ then a sufficient condition for

$$\frac{1}{c_p} \sim \frac{k}{N_{\max}}$$

to hold for large k for the lowest mode is that $N^2(t)/t$ is convex for $0 \leqslant t \leqslant 1$ where $t = \mathrm{e}^{-\beta x}$, and $1/\beta$ is an arbitrary length scale.

Finally it was demonstrated that fine-scale peaks and troughs in the $N^2(x)$ function are effectively smoothed by a mode 1 eigenfunction provided that the angular frequency of the wave is not too great.

The author thanks Gary Watson of the Mathematics Department, Bristol University, for much help in the preparation of this paper. Also thanked is Anton Edwards of the Dunstaffnage Marine Laboratory, Oban, Scotland for taking the measurements of the water column in Loch Linnhe on which figure 3 and subsequent figures are based. The photograph in figure 1 was taken by John Branson of DRA Farnborough.

REFERENCES

CHENEY, E. W. 1966 *Introduction to Approximation Theory*. McGraw-Hill.

ECKMAN, V. W. 1906 On dead water. In *The Norwegian North Polar Expedition 1893-1896, Scientific Results* (ed. F. Nansen), vol. V, chap. XV. Longmans, Green and Co.

ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. & TRICOMI, F. G. 1953 *Higher Transcendental Functions, vol. 1*. McGraw-Hill.

GARRETT, C. & MUNK, W. 1972 Space-time scales of internal waves. *Geophys. Fluid Dyn.* **2**, 225–264.

HARDY, G. H., LITTLEWOOD, J. E. & PÓLYA, G. 1959 *Inequalities*. Cambridge University Press.

KELLER, J. B. & MUNK, W. H. 1970 Internal wave wakes of a body moving in a stratified fluid. *Phys. Fluids* **13**, 1425–1431.

KRAUSS, W. 1966 *Methoden und Ergebnisse der Theoretischen Ozeanographie: Band II, Interne Wellen*. Gebrüder Borntraeger.

INCE, E. L. 1926 *Ordinary Differential Equations*. Longmans, Green and Co.

LEE-BAPTY, I. P. 1991 An equation to model internal-wave wakes in the region of the Mach cone. *Tech. Rep. RAE TR 91055*. Defence Research Agency, Farnborough, Hants, UK.

LEE-BAPTY, I. P. 1992 Structure of internal-wave wakes: linear theory. *Tech. Rep. DRA TR 92005*. Defence Research Agency, Farnborough, Hants, UK.

LIGHTHILL, M. J. 1978 *Waves in Fluids*. Cambridge University Press.

MILOH, T. & TULIN, M. P. 1988 Dead water phenomena: a non-linear theory of wave disturbances due to a ship in a shallow thermocline. *Tech. Rep. 88-29*. Ocean Engng Lab., University of California, Santa Barbara, USA.

PERRY, J. R. 1992 Experimental results in SAR imaging of ship wakes. *Technical Report DRA TR 92044, Defence Research Agency, Farnborough, Hants, UK*.

PHILLIPS, O. M. 1977 *The Dynamics of the Upper Ocean*. Cambridge University Press.

PRESS, W. H., FLANNERY, B. P., TEUKOLSKY, S. A. & VETTERLING, W. T. 1986 *Numerical Recipes*. Cambridge University Press.

SLATER, L. J. 1960 *Confluent Hypergeometric Functions*. Cambridge University Press.

STAPLETON, N. R. & PERRY, J. R. 1992 Synthetic aperture radar imagery of ship generated internal waves during the UK-US Loch Linnhe series of experiments. *12th Annual Intl Geoscience and Remote Sensing Symp. (IGARSS '92), Houston, Texas, USA, 1992*, pp. 1338-1340.

TRICOMI, F. G. 1954 *Funzioni Ipergeometriche Confluenti*. Edizioni Cremonese.

WATSON, G., CHAPMAN, R. D. & APEL, J. R. 1992 Measurements of the internal wave wake of a ship in a highly stratified sea loch. *J. Geophys. Res.* **97**, C6, 9689-9703.

WATSON, G. N. 1944 *Theory of Bessel Functions*. Cambridge University Press.

WHITTAKER, E. T. 1918 A formula for the solution of algebraic or transcendental equations. *Proc. Edinburgh Math. Soc.* **36**, 103-106.

YIH, C.-S. 1965 *Dynamics of Nonhomogenous Fluids*. Macmillan.